不連続ガレルキン時間離散化手法の変分法的な解析

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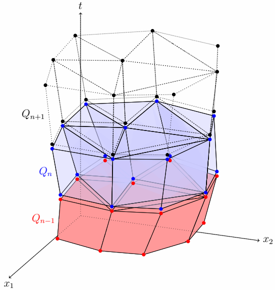
Time discretizations of parabolic equations

Example: $\partial_t u - \Delta u = f$ in $\Omega \times (0, T)$, $u|_{\partial\Omega} = 0$, $u|_{t=0} = u_0$.

- The θ method (forward Euler, backward Euler and Crank–Nicolson, ...);
- Explicit/implicit Runge–Kutta method;
- Multi-step methods;
- Padé approximation (rational approximation of $e^{t\Delta}$)

They essentially compute approximations at nodal points $t_n = n\Delta t$.



(Thanks to K. Takizawa)

Discontinuous Galerkin (DG) time-stepping method

- Approximate functions are piece-wise polynomials in t;
- The method is well applied to Space-Time Computation Technique.
- The method is based on Lions' weak formulation.

Heat equation

$\Omega \subset \mathbb{R}^d$: bounded domain with $d \geq 1$, $\quad T > 0$. $\qquad (\partial_t u = \frac{\partial u}{\partial t})$

Given $f = f(x,t)$ and $u_0 = u_0(x)$, find $u = u(x,t)$ such that

$$
\begin{aligned}
\partial_t u - \Delta u &= f, && x \in \Omega,\ t \in (0,T),\\
u &= 0, && x \in \partial\Omega,\ t \in (0,T),\\
u(x,0) &= u_0(x), && x \in \Omega.
\end{aligned}
$$

Approaches for the well-posedness:

- Several "classical" methods....
- Semigroup theory $\quad$ (C_0, analytical)
- Maximal regularity
- Variational method of J. L. Lions

linear vs. nonlinear

The following discussion remains valid for more general "parabolic" PDEs.

Weak formulation

- $\mathcal{X} = \{v \in L^2(0,T;H_0^1) \mid \partial_t v \in L^2(0,T;H^{-1})\}$, $\quad \mathcal{Y} = L^2(0,T;H_0^1) \times L^2$
- $\|v\|_{\mathcal{X}}^2 = \|\partial_t v\|_{L^2(0,T;H^{-1})}^2 + \|v\|_{L^2(0,T;H_0^1)}^2$, $\quad \|v\|_{\mathcal{Y}}^2 = \|v_1\|_{L^2(0,T;H_0^1)}^2 + \|v_2\|^2$
- **Remark.** $\mathcal{X} \subset C^0([0,T];L^2)$.

Find $u \in \mathcal{X}$ such that

$$\underbrace{\int_0^T \int_\Omega [(\partial_t u)v_1 + \nabla u \cdot \nabla v_1] \ dxdt + \int_\Omega u(0)v_2 \ dx}_{=B(u,v)} = \underbrace{\int_0^T \int_\Omega f v_1 \ dxdt + \int_\Omega u_0 v_2 \ dx}_{=L(v)}$$

$$\forall v = (v_1, v_2) \in \mathcal{Y}.$$

Lions' Theorem

There exists a unique $u \in \mathcal{X}$ and $\|u\|_{\mathcal{X}} \leq C(\|u_0\| + \|f\|_{L^2(0,T;H^{-1})})$.

- **Standard proof.** Galerkin approximation, $V_N = \mathrm{span}\{\phi_i\}_{i=1}^N \subset V$ and so on
- **Alternate proof.** Application of Banach–Nečas–Babuška's Theorem

関数空間とノルムなど

Hilbert 空間

$$L^2 = L^2(\Omega), \quad (w,v) = \int_\Omega w(x)v(x)\ dx, \quad \|w\|^2 = (w,w) = \int_\Omega w(x)^2\ dx$$

$$H_0^1 = H_0^1(\Omega) = \{w \in L^2 \mid \nabla w \in (L^2)^d,\ v|_\Gamma = 0\},$$

$$(\nabla w, \nabla v) = \int_\Omega \nabla w \cdot \nabla v\ dx = \int_\Omega \sum_{j=1}^{d} (\partial_j w)(\partial_j v)\ dx, \quad \|w\|_{H_0^1}^2 = \int_\Omega |\nabla w|^2\ dx$$

$$H^{-1} = H^{-1}(\Omega) = [H_0^1]' = \{\varphi : H_0^1 \to \mathbb{R} \text{ linear } \mid \varphi(v)/\|v\|_{H_0^1} < \infty\}$$

$$\|\varphi\|_{H^{-1}} = \sup_{v \in H_0^1} \frac{\varphi(v)}{\|v\|_{H_0^1}} \qquad \text{(普通，} \langle\varphi, v\rangle = \varphi(v) \text{ と書く.)}$$

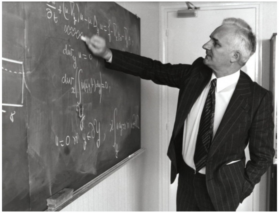
L^2 とその双対空間 $(L^2)'$ を同一視し，Gelfand triple を考える：

$$H_0^1 \hookrightarrow L^2 \simeq (L^2)' \hookrightarrow H^{-1} \qquad \text{(埋め込みは稠密で連続).}$$

Bochner 空間

$$X = L^2, H_0^1, H^{-1}$$

$$v \in L^2(0,T;X) \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \|v\|_{L^2(0,T;X)}^2 = \int_0^T \|v\|_X^2\ dt < \infty.$$



Jacques-Louis Lions (1928–2001)

http://www.breves-de-maths.fr/

"......I recall a remark of Jacques-Louis LIONS that a framework which is too general cannot be very deep, and he had made this comment about semigroup theory; he did not deny that the theory is useful, and the proof of the Hille–Yosida theorem is certainly more easy to perform in the abstract setting of a Banach space than in each particular situation, but the result applies to equations with very different properties that the theory cannot distinguish......" (L. Tartar: *An Introduction to Sobolev Spaces and Interpolation Spaces*, Springer, 2007; page XIV)

Banach–Nečas–Babuška Theorem

- V: (real) Banach space; W: (real) reflexive Banach space
- a: bilinear form on $V \times W$, $\|a\| \overset{\text{def.}}{=} \sup_{v \in V, w \in W} \frac{a(v,w)}{\|v\|_V \cdot \|w\|_W} < \infty$ bounded

Theorem BNB

The following (i) , (ii) and (iii) are equivalent.

(i) For any $L \in W'$, there exists a unique $u \in V$ s.t.

$$a(u,w) = L(w) \quad (\forall w \in W).$$

(ii) The following (BNB1) and (BNB2) hold:

$$\exists \beta > 0, \quad \inf_{v \in V} \sup_{w \in W} \frac{a(v,w)}{\|v\|_V \|w\|_W} = \beta; \tag{BNB1}$$

$$w \in W, \quad (\forall v \in V,\ a(v,w) = 0) \quad \Longrightarrow \quad (w = 0). \tag{BNB2}$$

(iii) The following (BNB3) holds:

$$\exists \beta > 0, \quad \inf_{v \in V} \sup_{w \in W} \frac{a(v,w)}{\|v\|_V \|w\|_W} = \inf_{w \in W} \sup_{v \in V} \frac{a(v,w)}{\|v\|_V \|w\|_W} = \beta. \tag{BNB3}$$

Remarks

A priori estimate

$$u \in V\text{: the solution} \quad \Longrightarrow \quad \|u\|_V \leq \frac{1}{\beta}\|L\|_{W'}. \qquad \left[\ \|L\|_{W'} \overset{\text{def.}}{=} \sup_{w \in W} \frac{L(w)}{\|w\|_W}\ \right]$$

Alternate expression of (BNB1)

$$\exists \beta > 0, \quad \inf_{v \in V} \sup_{w \in W} \frac{a(v,w)}{\|v\|_V \|w\|_W} = \beta \qquad \text{(BNB1)}$$

is expressed as

$$\exists \beta > 0, \quad \sup_{w \in W} \frac{a(v,w)}{\|w\|_W} \geq \beta \|v\|_V \quad (\forall v \in V).$$

Alternate expression of (BNB2)

$$w \in W, \quad (\forall v \in V,\ a(v,w) = 0) \quad \Longrightarrow \quad (w = 0) \qquad \text{(BNB2)}$$

is expressed as

$$\sup_{v \in V} |a(v,w)| > 0 \quad (\forall w \in W, w \neq 0).$$

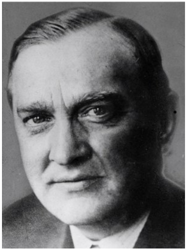
Long history....

- As for the naming of "BNB", I follow Ern & Guermond 2004, FEM book

  Other namings: Generalized Lax–Milgram, Babuška–Lax–Milgram, ...

  In Ern & Guermond 2002, they called it Nečas' Theorem
- Nečas 1962, 1967 "(iii) $\Rightarrow$ (i)" and "(ii)$'$ $\Rightarrow$ (i)", Hilbert space
- Babuška 1971 "(iii) $\Rightarrow$ (i)", Hilbert space
- Babuška and Aziz 1972 "(ii) $\Rightarrow$ (i)", Hilbert space
- BNB theorem is a re-phrasing of OMT (CGT) and CRT of Banach.
- Brezzi 1974 "(iii) $\Leftrightarrow$ (i)", Hilbert space
- Roşca 1989 "(ii) $\Leftrightarrow$ (i)", Banach space
- Application (or another origin): **Mixed finite element method.**

  Brezzi 1974, Kikuchi 1973 (See Brezzi & Fortin 1991; Boffi, Brezzi & Fortin 2013)
- Application: **Proof of Lions' theorem** by showing (BNB1) and (BNB2). See Ern and Guermond 2004. (Schwab & Stevenson 2009)



Stefan Banach (1892–1945)

Born in Kraków, ...

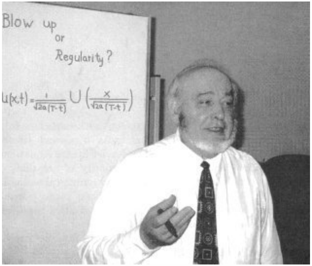
...... Around 1929 he began writing his *Théorie des opérations linéaires*.

...... In 1941, when the Germans took over Lwów, all institutions of higher education were closed to Poles. As a result, Banach was forced to earn a living as a feeder of lice at Rudolf Weigl's Institute for Study of Typhus and Virology.

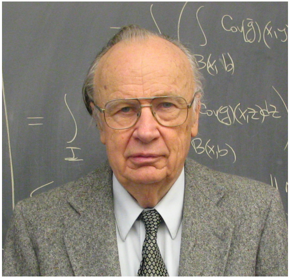
...... he died in August 1945, having been diagnosed seven months earlier with lung cancer.

...... Some of the notable mathematical concepts that bear Banach's name include Banach spaces, Banach algebras, the Banach–Tarski paradox, the Hahn–Banach theorem, the Banach–Steinhaus theorem, the Banach–Mazur game, the Banach–Alaoglu theorem, and the Banach fixed-point theorem.

`https://en.wikipedia.org/wiki/Stefan_Banach`



Jindřich Nečas (1929–2002)



Ivo M. Babuška (1926–)

http://msekce.karlin.mff.cuni.cz/memories/necas/
https://users.ices.utexas.edu/~babuska/babuska_homepage/photos.html

"Mathematics Genealogy Project" https://www.genealogy.math.ndsu.nodak.edu/

- Babuška: Ph.D. 1955 Advisors: Eduard Čech, Vladimir Knichal
- Nečas: Ph.D. 1956 Advisor: Ivo M. Babuška

Nečas was the first Ph.D. student of Babuška.
http://www.karlin.mff.cuni.cz/memories/necas/

Lions' Theorem and BNB Theorem

Weak formulation of Heat equation

Find $u \in \mathcal{X}$ such that $B(u, v) = L(v)$ for all $v = (v_1, v_2) \in \mathcal{Y}$.

Lemma

$$\exists \beta > 0, \quad \inf_{w \in \mathcal{X}} \sup_{v \in \mathcal{Y}} \frac{B(w, v)}{\|w\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}} = \beta; \tag{BNB1}$$

$$v \in \mathcal{Y}, \quad (\forall w \in \mathcal{X},\ B(w, v) = 0) \quad \Longrightarrow \quad (v = 0). \tag{BNB2}$$

According to Lemma, we can apply BNB theorem to conclude $\exists 1 u \in \mathcal{X}$ and $\|u\|_{\mathcal{X}} \le C \|L\|_{\mathcal{Y}'}$ (Lions' Theorem!).

How to prove (BNB1)? Set $v = ((-\Delta)^{-1}\partial_t w + \mu w, \mu w(0)) \in \mathcal{Y}$ for $w \in \mathcal{X}$ and $\mu > 0$. Then, for large μ, we can prove

$$\|v\|_{\mathcal{Y}} \le C_1 \|w\|_{\mathcal{X}}, \qquad B(w, v) \ge C_2 \|w\|_{\mathcal{X}}^2$$

using

$$\langle g, (-\Delta)^{-1} g \rangle \ge C_3 \|g\|_{H^{-1}}^2, \quad \|(-\Delta)^{-1} g\|_{H_0^1} \le C_4 \|g\|_{H^{-1}} \quad (g \in H^{-1}).$$

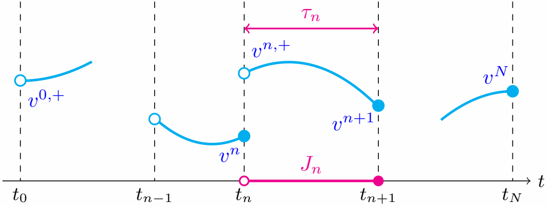
See Ern and Guermond 2004 or Saito 2017 (Notes on ...).

Partition Δ_τ of $J = (0, T)$

- $0 = t_0 < t_1 < \cdots < t_N = T, \quad J_n = (t_n, t_{n+1}], \quad \tau_n = t_{n+1} - t_n$
- $\Delta_\tau = \{J_0, \ldots, J_{N-1}\}, \quad \tau = \max_{0 \le n \le N-1} \tau_n$

$$C^0(\Delta_\tau; H) = \{v \in L^\infty(J; H) \mid v|_{J_n} \in C^0(J_n; H),\ 0 \le n \le N-1\}$$

$$v^{n,+} = \lim_{t \downarrow t_n} v(t), \quad v^{n+1} = v(t_{n+1}).$$



For integer $q \ge 0$,

$$\mathcal{S}_\tau = \mathcal{S}_\tau^q(H, V) = \{v \in C^0(\Delta_\tau; H) \mid v|_{J_n} \in \mathcal{P}^q(J_k; V),\ 0 \le n \le N-1\}$$

DG time-stepping method

dG(q) method

Find $u_\tau \in \mathcal{S}_\tau$ such that

$$\underbrace{\sum_{n=0}^{N-1} \int_{J_n} [(\partial_t u_\tau, v) + (\nabla u_\tau, \nabla v)] \ dt + (u_\tau^{0,+}, v^{0,+}) + \sum_{n=1}^{N-1} (u_\tau^{n,+} - u_\tau^{n}, v^{n,+})}_{=B_\tau(u_\tau, v)}$$

$$= \int_J (f, v) \ dt + (u_0, v^{0,+}) \qquad (\forall v \in \mathcal{S}_\tau),$$

- Lasaint & Raviart 74 for ODE, Jamet 78 for PDE in moving regions
- See Thomée 07 for a derivation
- Hulme 72a, 72b $\cdots$ similar DG methods

Lemma (Consistency)

$u \in \mathcal{X}$: solution of the heat equation, $u_\tau \in \mathcal{S}_\tau$: solution of dG($q$)

$$\Rightarrow \qquad B_\tau(u - u_\tau, v) = 0 \quad (\forall v \in \mathcal{S}_\tau).$$

Remarks

dG(q) method: Supposing u_τ^n is given, we find u_τ in $J_n = (t_n, t_{n+1}]$ by solving:

$$\int_{t_n}^{t_{n+1}} [(\partial_t u_\tau, \phi) + (\nabla u_\tau, \nabla \phi)] + (u_\tau^{n,+} - u_\tau^n, \phi^{n,+}) = \int_{t_n}^{t_{n+1}} (f, \phi) \quad (\forall \phi \in \mathcal{P}^q(J_n; V)).$$

- **$q = 0$:** Set $u_\tau|_{J_n} = U$ for $U \in V$.

$$U - \tau_n \Delta U = u_\tau^n + \int_{J_n} f \quad \Leftrightarrow \quad \frac{U - u_\tau^n}{\tau_n} - \Delta U = \frac{1}{\tau_n} \int_{J_n} f.$$

  dG(0): backward Euler

- **$q = 1$:** Set $u_\tau|_{J_n} = U_0 + U_1 \frac{t - t_n}{\tau_n}$ for $U_0, U_1 \in V$.

$$(I - \tau_n \Delta) U_0 + (I - \frac{\tau_n}{2} \Delta) U_1 = u_\tau^n + \int_{J_n} f,$$

$$-\frac{\tau_n}{2} \Delta U_0 + (\frac{1}{2} I - \frac{\tau_n}{3} \Delta) U_1 = \frac{1}{\tau_n} \int_{J_n} (t - t_n) f$$

  Moreover, if $f = 0$,

$$u_\tau^{n+1} = U_0 + U_1 = \left(I - \frac{2}{3} \tau_n \Delta + \frac{\tau_n^2}{6} \Delta^2 \right)^{-1} \left(I + \frac{\tau_n}{3} \Delta \right) u_\tau^n \;=\; R_{2,1}(\tau_n \Delta) u_\tau^n$$

  dG(1) ,$f = 0$: sub-diagonal $(2, 1)$ Padé rational approximation of e^{-z}

Previous studies

- Lasaint & Raviart 74 — dG(q) for ODE and analysis for 1st order PDE
  - dG(q), $f = 0$: sub-diagonal $(q+1, q)$ Padé rational approximation of e^{-z}
  - dG(q): implicit Runge–Kutta, strongly A-stable of order $2q+1$
- Jamet 78 — $\Omega(t)$, error estimates in $L^2(H^1)$ and $L^\infty(L^2)$
- Eriksson, Johnson & Thomée 85 — error estimates in $L^\infty(L^2)$, super-convergence
- Thomée 06 — error estimates in $L^\infty(L^2)$, super-convergence
- Eriksson & Johnson 91, 95 — $q = 0, 1$, error estimates in $L^\infty(L^2)$, $L^\infty(L^\infty)$ with log-factor
- Chrysafinos & Walkington 06 — error estimates using projections
- Leykekhman & Vexler 16 — best approximation in $L^\infty(L^\infty)$ with log-factor
- (Kemmochi 17 — $f = 0$, $L^\infty(L^p)$ error estimates, rational approximation)

Norms (1/2)

$$\left\{\begin{array}{c}\|u\|^2_{\mathcal{X},\tau} \\ \|u\|^2_{\mathcal{X},\tau,\star} \\ \|u\|^2_{\mathcal{X},\tau,\#}\end{array}\right\} = \sum_{n=0}^{N-1} \int_{J_n} \left[\|\partial_t u\|^2_{H^{-1}} + \|u\|^2_{H^1_0}\right] \, dt + \|u^{0,+}\|^2 + \sum_{n=1}^{N-1} \left\{\begin{array}{c}1 \\ \tau_n^{-1} \\ \tau_n\end{array}\right\} \|u^{n,+} - u^n\|^2$$

$$\left\{\begin{array}{c}\|v\|^2_{\mathcal{Y},\tau} \\ \|v\|^2_{\mathcal{Y},\tau,\star} \\ \|v\|^2_{\mathcal{Y},\tau,\#}\end{array}\right\} = \sum_{n=0}^{N-1} \int_{J_n} \|v\|^2_{H^1_0} \, dt + \|v^{0,+}\|^2 + \sum_{n=1}^{N-1} \left\{\begin{array}{c}1 \\ \tau_n^{-1} \\ \tau_n\end{array}\right\} \|v^{n,+}\|^2.$$

Recall:

$$B_\tau(u,v) = \sum_{n=0}^{N-1} \int_{J_n} [(\partial_t u, v) + (\nabla u, \nabla v)] \, dt + (u^{0,+}, v^{0,+}) + \sum_{n=1}^{N-1} (u^{n,+} - u^n, v^{n,+}).$$

Lemma (boundness)

$$\|u\|_{\mathcal{X},\tau,\#} \le \|u\|_{\mathcal{X},\tau} \le \|u\|_{\mathcal{X},\tau,\star}, \qquad |B_\tau(u,v)| \le \begin{cases} M_0 \|u\|_{\mathcal{X},\tau} \|v\|_{\mathcal{Y},\tau} \\ M_1 \|u\|_{\mathcal{X},\tau,\star} \|v\|_{\mathcal{Y},\tau,\#} \\ M_2 \|u\|_{\mathcal{X},\tau,\#} \|v\|_{\mathcal{Y},\tau,\star}. \end{cases}$$

Norms (2/2)

$$\left\{\begin{array}{c} |||u|||^2_{\mathcal{Y},\tau} \\ |||u|||^2_{\mathcal{Y},\tau,\star} \\ |||u|||^2_{\mathcal{Y},\tau,\#} \end{array}\right\} = \sum_{n=0}^{N-1} \int_{J_n} \|u\|^2_{H^1_0} \, dt + \|u^N\|^2 + \sum_{n=1}^{N-1} \left\{\begin{array}{c} 1 \\ \tau_n^{-1} \\ \tau_n \end{array}\right\} \|u^{n,+}\|^2,$$

$$\left\{\begin{array}{c} |||v|||^2_{\mathcal{X},\tau} \\ |||v|||^2_{\mathcal{X},\tau,\star} \\ |||v|||^2_{\mathcal{X},\tau,\#} \end{array}\right\} = \sum_{n=0}^{N-1} \int_{J_n} \left[\|\partial_t v\|^2_{H^{-1}} + \|v\|^2_{H^1_0}\right] \, dt + \|v^N\|^2 + \sum_{n=1}^{N-1} \left\{\begin{array}{c} 1 \\ \tau_n^{-1} \\ \tau_n \end{array}\right\} \|v^{n,+} - v^n\|^2.$$

An alternate expression of $B_\tau(u, v)$:

$$B_\tau(u, v) = \sum_{n=0}^{N-1} \int_{J_n} [-(u, \partial_t v) + (\nabla u, \nabla v)] \, dt + (u^N, v^N) + \sum_{n=1}^{N-1} (u^n, v^n - v^{n,+}).$$

Lemma (boundness)

$$|||u|||_{\mathcal{Y},\tau,\#} \leq |||u|||_{\mathcal{Y},\tau} \leq |||u|||_{\mathcal{Y},\tau,\star}, \qquad |B_\tau(u, v)| \leq \begin{cases} M_0' |||u|||_{\mathcal{Y},\tau} |||v|||_{\mathcal{X},\tau} \\ M_1' |||u|||_{\mathcal{Y},\tau,\star} |||v|||_{\mathcal{X},\tau,\#} \\ M_2' |||u|||_{\mathcal{Y},\tau,\#} |||v|||_{\mathcal{X},\tau,\star}. \end{cases}$$

BNB inequalities and best approximations

Theorem A (S 2017)

For an integer $q \geq 1$, $\quad \exists c_1,\ c_2 > 0$ s.t.

$$\inf_{w \in \mathcal{S}_\tau} \sup_{v \in \mathcal{S}_\tau} \frac{B_\tau(w, v)}{\|w\|_{\mathcal{X},\tau} \|v\|_{\mathcal{Y},\tau,\#}} = c_1, \qquad \inf_{w \in \mathcal{S}_\tau} \sup_{v \in \mathcal{S}_\tau} \frac{B_\tau(w, v)}{|||w|||_{\mathcal{Y},\tau,\#} |||v|||_{\mathcal{X},\tau}} = c_2.$$

For any $w \in \mathcal{S}_\tau$, we have

$$\begin{aligned} \|w - u_\tau\|_{\mathcal{X},\tau} &\leq \frac{1}{c_1} \sup_{v_\tau \in \mathcal{S}_\tau} \frac{B_\tau(w - u_\tau, v_\tau)}{\|v_\tau\|_{\mathcal{Y},\tau,\#}} = \frac{1}{c_1} \sup_{v_\tau \in \mathcal{S}_\tau} \frac{B_\tau(w - u, v_\tau)}{\|v_\tau\|_{\mathcal{Y},\tau,\#}} \\ &\leq \frac{M_1}{c_1} \|w - u\|_{\mathcal{X},\tau,\star}. \end{aligned}$$

Theorem B (S 2017)

Letting $u \in \mathcal{X}$: sol of heat equation & $u_\tau \in \mathcal{S}_\tau$: sol of dG($q$) for $q \geq 1$, we have

$$\|u - u_\tau\|_{\mathcal{X},\tau} \leq \left(1 + \frac{M_1}{c_1}\right) \inf_{w \in \mathcal{S}_\tau} \|u - w\|_{\mathcal{X},\tau,\star},$$

$$|||u - u_\tau|||_{\mathcal{Y},\tau,\#} \leq \left(1 + \frac{M_2'}{c_2}\right) \inf_{w \in \mathcal{S}_\tau} |||u - w|||_{\mathcal{Y},\tau}.$$

Optimal order error estimates

The Lagrange interpolation $I_n v \in \mathcal{P}^q(\overline{J_n}; H_0^1)$ is defined in a usual way for $v \in L^2(J_n; H_0^1) \cap H^1(J_n; H^{-1})$. Using Taylor's Theorem, we deduce

Lemma

Letting $q \geq 1$, then there exists an absolute positive constant C such that

$$\|v - I_n v\|_{L^2(J_n;U)} \leq C\tau_n^s \|v^{(s)}\|_{L^2(J_n;U)},$$
$$\|v' - (I_n v)'\|_{L^2(J_n;U)} \leq C\tau_n^{s-1} \|v^{(s)}\|_{L^2(J_n;U)}$$

for $1 < s \leq q+1$ and $v \in H^{q+1}(J_n; U)$, where $U = H_0^1, L^2, H^{-1}$; C is independent of U.

Theorem C (S 2017)

Let $q \geq 1$. If u is suitably smooth, we have:

$$\left(\sum_{n=0}^{N-1} \|\partial_t u - \partial_t u_\tau\|_{L^2(J_n;H^{-1})}\right)^{1/2} \leq c_3 \tau^q \left(\|\partial_t^{q+1} u\|_{L^2(J;H^{-1})} + \|\partial_t^q u\|_{L^2(J;H_0^1)}\right);$$
$$\sup_{1\leq n\leq N} \|u(t_n) - u_\tau(t_n)\| + \|u - u_\tau\|_{L^2(J;H_0^1)} \leq c_4 \tau^{q+1} \|\partial_t^{q+1} u\|_{L^2(J;H_0^1)}.$$

Sketch of the proof of BNB inequalities (1/2)

$$(*) \qquad \inf_{w\in\mathcal{S}_\tau} \sup_{v\in\mathcal{S}_\tau} \frac{B_\tau(w,v)}{\|w\|_{\mathcal{X},\tau}\|v\|_{\mathcal{Y},\tau,\#}} = c_1$$

It suffices to prove, $\forall w \in \mathcal{S}_\tau$, $\exists v \in \mathcal{S}_\tau$ s.t.

$$(\#) \qquad B_\tau(w,v) \geq C\|w\|^2_{\mathcal{X},\tau} \quad \text{and} \quad \|v\|_{\mathcal{Y},\tau,\#} \leq C\|w\|_{\mathcal{X},\tau}.$$

Indeed, v is given as $v = \Pi(-\Delta)^{-1}\partial_t w + \mu w$ with large μ, where

- $\Pi\phi \in \mathcal{S}_\tau$ for $\phi \in L^2(J;V)$ is defined as $\phi_n = (\Pi\phi)|_{J_n}$, where

$$(\phi_n)(t^{n,+}) = 0, \quad \int_{J_n} (\phi_n, \chi)\, dt = \int_{J_n} (\phi, \chi)\, dt \quad (\forall \chi \in \mathcal{P}^{q-1}(J_n, V))$$

- We have $\|\phi_n\|_{L^2(J_n,V)} \leq C\|\phi\|_{L^2(J_n,V)}$.

Following the continuous case, we are able to deduce $(\#)$ using

$$\langle g, (-\Delta)^{-1} g\rangle \geq C_3\|g\|^2_{H^{-1}}, \quad \|(-\Delta)^{-1}g\|_{H^1_0} \leq C_4\|g\|_{H^{-1}} \quad (g \in H^{-1})$$

and

$$(\chi^{n,+} - \chi^n, \chi^{n,+}) = \frac{1}{2}\|\chi^{n,+}\|^2 - \frac{1}{2}\|\chi^n\|^2 + \frac{1}{2}\|\chi^{n,+} - \chi^n\|^2 \quad (\chi \in \mathcal{S}_\tau).$$

Sketch of the proof of BNB inequalities (2/2)

$$(**) \qquad \inf_{w \in \mathcal{S}_\tau} \sup_{v \in \mathcal{S}_\tau} \frac{B_\tau(w, v)}{|||w|||_{\mathcal{Y},\tau,\#} |||v|||_{\mathcal{X},\tau}} = c_2$$

The direct proof of $(**)$ is apparently so difficult that we take a detour.
We apply the equivalence "(ii)$\Leftrightarrow$(iii)" in BNB Theorem.
That is, it suffices to prove

$$\text{(a)} \qquad \exists c_2 > 0, \quad \inf_{v_\tau \in \mathcal{S}_\tau} \sup_{w_\tau \in \mathcal{S}_\tau} \frac{B_\tau(w_\tau, v_\tau)}{|||w_\tau|||_{\mathcal{Y},\tau,\#} |||v_\tau|||_{\mathcal{X},\tau}} = c_2;$$

$$\text{(b)} \qquad v_\tau \in \mathcal{S}_\tau, \quad (\forall w_\tau \in \mathcal{S}_\tau, \; B_\tau(w_\tau, v_\tau) = 0) \quad \Longrightarrow \quad (v_\tau = 0).$$

In fact,

- (b) follows $(*)$.
- (a) could be obtained in exactly the same way as the derivation of $(*)$.

Q.E.D.

まとめ

- 熱方程式に対する DG time-stepping 法について，BNB Theorem に基づいた，変分法的な解析を遂行した．放物型をあたかも「楕円型のように」扱え，結果として最適誤差評価が得られる．
- もっと一般の，時空間に依存する係数を持つ放物型問題についても，解析と結果は全く同じである．Saito 2017 (Variational ...) を参照されたい．
- 時間離散化のみ扱ったが，例えば，空間変数を有限要素法で離散化した，全離散スキームを考えても，時空間での最適誤差評価が得られる．このとき，仮定する解の正則性も最適であることは重要である．Saito 2017 (Variational ...) を参照されたい．
- 今後の展開：
  - いろいろな空間離散化との組み合わせ（「安定性」の問題）
  - 非斉次 Dirichlet 境界条件に対する Nitsche 法の変分法的解析（上田）
  - dG(q) 法の有理関数近似としての解析（剱持）

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Thank you for your attention!

This slide is available at http://www.infsup.jp/saito/

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附録について

- §6 附録：楕円型方程式に対する DG 法
  講義では時間変数に関する不連続 Galerkin (DG) 法を扱った．この附録では，空間変数に関する DG 法の例を簡単に紹介する．ただし，記号等の説明はしていないので，あまり役に立たないかもしれない．参考書としては，

  B. M. Rivière,
  Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation, SIAM, 2008

  を挙げておく．
- §7 附録：加藤敏夫と数値解析
  RIMS 研究集会（2017 年 11 月 9 日）で講演した際に，BNB 定理の証明においては，加藤敏夫先生の 1958 年の論文の手法を応用すると，より明快で理解が深まる，という私見を述べた．詳しくは，Saito 2017 (Notes on ...) を参照されたい．また，加藤敏夫先生の数値解析に関するご業績を簡単に紹介した．今回の講義ではその点には触れなかったが，有益な情報なので，RIMS 研究集会に参加していなかった人たちのために，附録としてその際の資料を残したい．

Lax–Milgram Theorem

- V: (real) Hilbert space $\quad (\cdot,\cdot), \quad \|\cdot\|$
- V': the dual space of V $\quad$ (the set of all bounded linear forms on V)
- a: bilinear form on $V \times V$, $\qquad \|a\| \stackrel{\text{def.}}{=} \sup_{u,v\in V} \dfrac{a(u,v)}{\|u\|\cdot\|v\|} < \infty$ bounded

Theorem LM

If
$$\exists \alpha > 0, \qquad a(v,v) \geq \alpha \|v\|^2 \quad (v \in V),$$
then, for any $L \in V'$, there exists a unique $u \in V$ s.t.
$$a(u,v) = L(v) \quad (\forall v \in V).$$

Remark. $\quad \|u\| \leq \dfrac{1}{\alpha}\|L\|_{V'}.$ $\qquad \left[\|L\|_{V'} \stackrel{\text{def.}}{=} \sup_{v\in V} \dfrac{L(v)}{\|v\|} \right]$

Applications

1. **Poisson equation.** $-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$ (bdd domain), $u = 0$ on $\partial\Omega$.

$$V = H_0^1(\Omega), \quad a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad L(v) = \int_\Omega fv \, dx.$$

2. Diffusion–convection–reaction equations, Bi-Laplace equations, and *many*
3. **Galerkin approximation.** $V_h \subset V$: subspace, $\quad \dim V_h (\sim h^{-d}) < \infty$

$$u_h \in V_h, \quad a(u_h, v_h) = L(v_h) \quad (\forall v_h \in V_h).$$

Galerkin orthogonality $a(u - u_h, v_h) = 0$ for all $v_h \in V_h$.

Céa's lemma $\|u - u_h\| \leq \dfrac{\|a\|}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|$

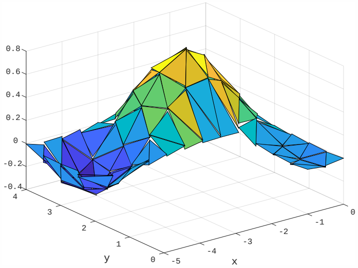
$$\begin{aligned} \alpha \|u - u_h\|^2 &\leq a(u - u_h, u - u_h) \\ &= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \qquad \forall v_h \in V_h \\ &\leq \|a\| \cdot \|u - u_h\| \cdot \|u - v_h\|. \end{aligned}$$

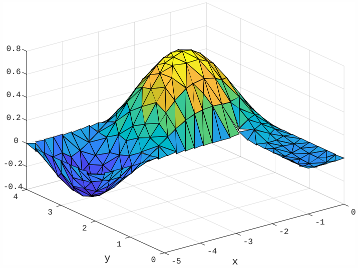
Discontinuous Galerkin (DG) method

- **Poisson equation.** $u \in V, \quad a(u,v) = L(v) \quad (\forall v \in V).$
- **SIP-DG method.** $u_h \in V_h, \quad a_h(u_h, v_h) = L(v_h) \quad (\forall v_h \in V_h).$

$$a_h(u,v) = \sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v)_{L^2(K)} - \sum_{e \in \mathcal{E}} (\langle\!\langle n \cdot \nabla u \rangle\!\rangle, [\![v]\!])_{L^2(e)}$$

$$- \sum_{e \in \mathcal{E}} ([\![u]\!], \langle\!\langle n \cdot \nabla v \rangle\!\rangle)_{L^2(e)} + \sum_{e \in \mathcal{E}} \frac{\beta}{h} ([\![u]\!], [\![v]\!])_{L^2(e)}$$





Error analysis of DG method

- **Poisson equation.** $u \in V, \quad a(u,v) = L(v) \quad (\forall v \in V)$.
- **SIP-DG method.** $u_h \in V_h, \quad a_h(u_h, v_h) = L(v_h) \quad (\forall v_h \in V_h)$.

Difficulty. $V_h \not\subset V \qquad (V_h = \{v \in L^\infty(\Omega) \mid v|_T \in \mathcal{P}^k(T)\ \forall T\} \not\subset V = H_0^1(\Omega))$.

Lemma

- (consistency) $a_h(u, v_h) = L(v_h) \quad \text{and} \quad a_h(u - u_h, v_h) = 0 \quad (\forall v_h \in V_h)$.
- (boundedness) $\exists M > 0, \quad a_h(w_h, v_h) \le M \|w_h\|_{\mathrm{DG}} \|v_h\|_{\mathrm{DG}} \quad (\forall w_h, v_h \in V_h)$.
- (coercivity) $\exists \alpha > 0, \quad a_h(v_h, v_h) \ge \alpha \|v_h\|_{\mathrm{DG}}^2 \quad (\forall v_h \in V_h)$.

$$
\begin{aligned}
\forall v_h \in V_h, \quad \alpha \|v_h - u_h\|_{\mathrm{DG}}^2 &\le a_h(v_h - u_h, v_h - u_h) \\
&= a_h(v_h - u, v_h - v_h) + a_h(u - u_h, v_h - u_h) \\
&\le M \cdot \|v_h - u\|_{\mathrm{DG}} \cdot \|v_h - v_h\|_{\mathrm{DG}}.
\end{aligned}
$$

Consequently, $\|u - u_h\|_{\mathrm{DG}} \le \left(1 + \dfrac{M}{\alpha}\right) \inf_{v_h \in V_h} \|u - v_h\|_{\mathrm{DG}}$.

Peter Lax (1926–) in Tokyo, 1969

PHOTO

Arthur Norton Milgram (1912–1961)

...... He made contributions in functional analysis, combinatorics, differential geometry, topology, partial differential equations, and Galois theory. ....

`https://en.wikipedia.org/wiki/Peter_Lax`
`https://en.wikipedia.org/wiki/Arthur_Milgram`

P.D. Lax, A. N. Milgram, *Parabolic equations*,
Annals of Mathematics Studies **33** (1954) 167–190.

"The following theorem is a mild generalization of the Fréchet–Riesz Theorem on the representation of bounded linear functionals in Hilbert space."

Associating operators with $a(\cdot,\cdot)$

BNB theorem: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

(i) $\forall L \in W',\ \exists 1 u \in V$ s.t. $a(u,w) = L(w)\ (\forall w \in W)$.

(ii)
$$\exists \beta > 0, \quad \inf_{v\in V} \sup_{w\in W} \frac{a(v,w)}{\|v\|_V \|w\|_W} = \beta; \tag{BNB1}$$
$$w \in W, \quad (\forall v \in V,\ a(v,w) = 0) \implies (w = 0). \tag{BNB2}$$

(iii)
$$\exists \beta > 0, \quad \inf_{v\in V} \sup_{w\in W} \frac{a(v,w)}{\|v\|_V \|w\|_W} = \inf_{w\in W} \sup_{v\in V} \frac{a(v,w)}{\|v\|_V \|w\|_W} = \beta. \tag{BNB3}$$

Linear operators $A : V \to W'$ and $A' : W \to V'$

$$ {}_{W'}\langle Av, w\rangle_W = a(v,w) = {}_V\langle v, A'w\rangle_{V'} \qquad (v \in V,\ w \in W).$$

Minimum modulus of A and A'

$$\mu(A) = \inf_{v\in V} \frac{\|Av\|_{W'}}{\|v\|_V} = \left(\sup_{v\in V} \frac{\|v\|_V}{\|Av\|_{W'}}\right)^{-1} \quad \text{and} \quad \mu(A') = \inf_{w\in W} \frac{\|A'w\|_{V'}}{\|w\|_W},$$

Remark. $\mu(A) = \inf_{v\in V} \frac{1}{\|v\|_V} \sup_{w\in W} \frac{{}_{W'}\langle Av, w\rangle_W}{\|w\|_W} = \inf_{v\in V} \sup_{w\in W} \frac{a(v,w)}{\|v\|_V \|w\|_W}$

Proof of BNB

BNB Theorem (operator version): (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

(i) A is bijective of $V \to W'$;

(ii) $\mu(A) > 0$ (BNB1), $\mathcal{N}(A') = \{0\}$ (BNB2);

(iii) $\mu(A) = \mu(A') > 0$ (BNB3).

Remark. $\|Av\|_{W'} \geq \mu(A)\|v\|_V \quad (\forall v \in V), \qquad \|A'w\|_{V'} \geq \mu(A')\|w\|_W \quad (\forall w \in W)$

(ii)$\Leftrightarrow$(iii) is easy:

- $\mu(A) > 0 \Rightarrow \mathcal{N}(A) = \{0\} \Rightarrow \mu(A) = \|A^{-1}\|^{-1}_{W',V}$
- $\mu(A') > 0 \Rightarrow \mathcal{N}(A') = \{0\} \Rightarrow \mu(A') = \|(A')^{-1}\|^{-1}_{V',W}$

(i)$\Leftrightarrow$(ii) is *not* easy (for Banach case):

- $\mu(A) > 0 \Leftrightarrow \mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A)$ is closed ....Closed Graph Th (CGT) is used!!
- A: bijective $\Leftrightarrow \mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A) = W'$

  $\Leftrightarrow \mathcal{N}(A) = \{0\}$, $\mathcal{R}(A)$ closed, $\mathcal{N}(A')^{\perp} = \mathcal{R}(A) = W'$

  ....Closed Range Th (CRT) is used!!

Proof of BNB: Alternate approach

Reduced minimum modulus of A and A'

$$\gamma(A) = \inf_{v\in V} \frac{\|Av\|_{W'}}{\mathrm{dist}_V(v,\mathcal{N}(A))} \qquad \text{and} \qquad \gamma(A') = \inf_{w\in W} \frac{\|A'w\|_{V'}}{\mathrm{dist}_W(w,\mathcal{N}(A'))},$$

where $\mathrm{dist}_V(v,\mathcal{N}(A)) = \inf_{g\in\mathcal{N}(A)} \|v-g\|_V$.

Remark. $\mathcal{N}(A)=\{0\} \quad \Rightarrow \quad \mathrm{dist}_V(v,\mathcal{N}(A)) = \|v\|_V$ and $\gamma(A)=\mu(A)$.

Theorem

(I) $\gamma(A) > 0 \Leftrightarrow \mathcal{R}(A)$ is closed; (II) $\gamma(A)=\gamma(A')$.

Theorem was given by

[Kato58] Tosio Kato. *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Analyse Math., **6** 261–322, 1958.

Remarks. (1) A might be (unbounded) linear closed operator and $\mathcal{D}(A)$ needs not be densely defined.
(2) Closed Range Th for unbounded operators could be proved using (I) and (II).

Introduction of $\gamma(A)$ was not originally Kato's idea. However, ...

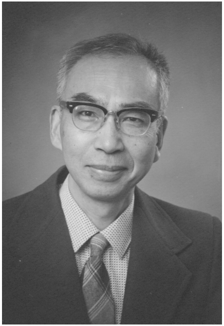
Proof of BNB: Alternate approach

R. G. Bartle stated in *Mathematical Review: MR0107819* that

> *The author introduces a constant $\gamma(A)$, called the lower bound of A, which is defined to be the supremum of all numbers $\gamma \geq 0$ such that $\|Ax\| \geq \gamma \|\tilde{x}\|$, $x \in D(A)$, where $\tilde{x}$ is the coset $x + N(A)$ and $\|\tilde{x}\|$ denotes the usual factor space norm in $X/N(A)$. Others have considered this constant before [cf. the reviewer's note, Ann. Acad. Sci. Fenn. Ser A. I. no. 257 (1958); MR0104172], but this reviewer is not aware of any previous systematic use of $\gamma(A)$.*

Small Notes:

- In [Kato58], he called $\gamma(A)$ the *lower bound of A*.
- Later, in his book ([Kato66, 76, 95] Perturbation theory...), he called $\gamma(A)$ the *reduced minimum modulus* of A, following [Gindler & Taylor 62] where $\mu(T)$ was defined.
- In [Kato58], he proved CRT for unbounded operators using $\gamma(A)$.
- In [Kato66, 76, 95], CRT is stated as a corollary of more general theorem; see also Brezis' book.



Tosio Kato (**加藤 敏夫** 1917–1999)

S.-T. Kuroda
T. Kato, T. Ikebe, H. Fujita, Y. Nakata

`http://www.ms.u-tokyo.ac.jp/~shu/Kato-2017-Conference.html`
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- MathSciNet: 13,701   Google Scholar: 18,700
- Tosio Kato's Method and Principle for Evolution Euqtions in Mathematical Physics (June 27–29, 2001, Sapporo)
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B. Simon, Tosio Kato's Work on Non–Relativistic Quantum Mechanics, arXiv:1711.00528 *(Outline:* arXiv:1710.06999*)*

Tosio Kato and Numerical Analysis

加藤敏夫
数理物理学（学生会員のために）
日本物理学会誌，第 15 巻，3 号，

1960 年，170-174

S　つまりプログラミングの問題ですか。

T　いやそれ以前の問題です。例えば偏微分方程式を digital な計算機で解こうとすれば，これを差分方程式でおきかえる他ありません。しかしこれはまことに乱暴な方法でして，それによつてどんな誤差がおこるか，誤差を小さくするにはどうしたらよいかということは，真面目に考えなければならない問題なのです。もつと簡単な一次方程式の解法とか，行列の固有値の計算などについても，同じような問題が残つています。この種の問題は近頃数値解析 (numerical analysis) と呼ばれていますが，これは応用数学の大きな分野になるでしょう。

S　それは数理物理より広い意味での応用数学でしょうね。

藤田宏：私の辿った道，
明治大学理工学部研究報告，特別寄稿，2000 年 3 月

加藤先生が最初に私に与えられたテーマは近似解法の研究であった。その頃、先生自身もシュレーディンガー作用素の固有値の摂動法を非有界な 2 次形式の理論と漸近近似の視点によって基礎づける研究を遂行しておられた。

当時、日本ではコンピュータはまだ稼動していないで、東大の物理教室の高橋秀俊先生と後藤英一君による PCI (パラメトロン計算機の 1 号機) と東大工学部の TAC がともに完成の間際であった。また、数値解法の専門家は日本には殆ど存在しなかった。こうした状況のもとで、私に近似解法の研究を指示されるにあたり、加藤先生は、「物理に籍をおいて数学をやると就職が難しい。その点で、将来普及するであろうコンピュータに用いる数学を勉強しておけば喰いはぐれがないだろう」と励まして下さった。それに従って、当時としては目新しかった差分法による偏微分方程式の近似解法を米、露の両スクールについて熱心に勉強した。

しかし、近似解法に関連して、私が最初のささやかな結果を得て物理学会のジャーナルに発表する事ができたのは、固有値の上下界や境界値問題の解の特定の点における値の上下界を導く、“上下界の理論”の応用であった（[1]、[2]）。加藤先生が編み出されたこ

（齊藤注：1952～1953 年頃の話と思われます.）

藤田先生の記事を読みたい方は，齊藤にご相談ください.

Tosio Kato and Numerical Analysis

- Kato–Temple formula for EVPs — Kato 1949
- Approximation theory for T^*T type operators
  Kato 1953, Fujita 1955, Fujita 1956, Kato et al. 1957
- Estimation for $\|A^n\|/\rho(A)^n$ — Kato 1960 (Numer. Math. **2**)
- Stability of FDM (When is von Neumann Condition sufficient for the stability?) — Kato 1960, Richtmyer & Morton 1957
- Kato–Trotter formula $\lim_{n\to\infty}[S_1(t/n)S_2(t/n)]^n = S(t)$, where $A = A_1 + A_2$
  Kato 1974 (linear), Kato & Masuda 1978 (nonlinear)
- Rational approximation of C_0 semigroup — Hersh & Kato 1979
- Indentity for projections: $P \neq 0, I$ and $P^2 = P \Rightarrow \|I - P\| = \|P\|$ Kato 1960
  See also Xu & Zikatanov 2003, Szyld 2006
- and many; "Perturbation theory" is closely related with Numerical Analysis.

参考書

摂動論と誤差解析は，解析の方法において共通点が多い．まだ，応用されていない良い方法があるかも．

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